1. If $\mathbf{A} = \begin{pmatrix} 3 & 5 & 3 \\ 1 & 7 & 3 \\ 1 & 2 & 8 \end{pmatrix}$ and $\mathbf{I}$ is a 3×3 identity matrix, what is the minimum value of λ such that the matrix $(\mathbf{A} - \lambda \mathbf{I})$ is singular?

2. Consider the following system of equations:

$$\begin{aligned} ax^2 + by^3 &= e \\ cx^2 + dy^3 &= f \end{aligned}$$

where a, b, c, d, e and f are real constants such that $ad \neq bc$. The maximum possible number of real solutions (x, y) of the system is

3. $\mathbf{A}$ is a 3×3 matrix with eigenvalues as 0, -1 and 1. What is the value of $\text{tr}(\mathbf{I} + \mathbf{A}^{100})$, where 'tr' denotes trace and $\mathbf{I}$ is an identity matrix?

4. Suppose $\mathbf{u} \in \mathbb{R}^{50 \times 1}$ is a unit vector with all non-zero elements. What is the maximum eigenvalue of $\mathbf{I} + 2\mathbf{u}\mathbf{u}^T$ if $\mathbf{I}$ is identity matrix?

5. Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. The vector $\mathbf{x}$ that minimizes $||A\mathbf{x} - \mathbf{b}||^2$, where $|| \cdot ||$ corresponds to the 2-norm (or Euclidean norm or magnitude) of a vector, is given as

6. If A is a 3×3 real skew-symmetric matrix (i.e. $A^T = -A$), and one of its eigen values is $-2j$, where $j = \sqrt{-1}$, the two other eigen values are given by

7. Find the value of k such that the system of equations is

$$x + ky = 1$$
$$kx + y = 1$$

has infinitely many solutions, when $k =$ ____.

8. For $A = \begin{bmatrix} 1+\cos(x) & \sin(x) & 0 \\ \sin(x) & 1+\cos(x) & 0 \\ 0 & 0 & 2+\cos(x) \end{bmatrix}$, the determinant of $A^\top (3A)^{-1}$ is

9. Find the rank of the matrix $\begin{bmatrix} 2 & 4 & 6 \\ 1 & 2 & 3 \\ 3 & 6 & 9 \end{bmatrix}$

10. Let $v_1 = [1; 1; a]$, $v_2 = [2; 2; 1]$ and $v_3 = [1; 2; 3]$. For what value of a does v_1 lie in the span of v_2 and v_3?

11. Let X be a random variable with probability density function

$$f(x) = \begin{cases} \frac{5}{8}(1 - x^4) & \text{if } -1 \leqslant x \leqslant 1 \\ 0 & \text{otherwise.} \end{cases}$$

If σ^2 denotes the variance of X, then what is the value of $3\sigma^2$?

12. If x, y and z are selected independently and at random from the interval $[0, 1]$, then the probability that $x \geqslant yz$ is

13. A person X fails to remember the last digit of the telephone number of his house physician Dr. Y. What is the probability that at least four attempts are necessary to find the actual number?

14. A girl alternatively tosses a fair coin and throws a die, starting with the coin. Find the probability that she will get a Tail before she gets a 4 or 5 on the die.

15. Let X and Y be two zero-mean continuous real-valued random variables with variances σ_X^2 and σ_Y^2 such that $\sigma_Y^2 > \sigma_X^2$. If the conditional probability density function of X given $Y = y$ is known as

$$f_{X|Y}(x|y) = \frac{1}{\sqrt{2\pi\sigma_X^2\left(1 - \frac{\sigma_X^2}{\sigma_Y^2}\right)}} \exp\left(-\frac{(x - \frac{\sigma_X^2}{\sigma_Y^2}y)^2}{2\sigma_X^2\left(1 - \frac{\sigma_X^2}{\sigma_Y^2}\right)}\right),$$

the value of $\int_{-\infty}^{\infty} x f_{X|Y}(x|y)dx$ for $\sigma_X^2 = 1$ and $\sigma_Y^2 = 2$ is

16. Suppose a fair die is rolled until the number 6 appears twice. The probability that the second 6 shows up on the fourth roll is

17. A certain disease affects 1% of a population. A test for the disease has the following probabilities:
- The probability of correctly identifying a diseased person is 0.95 (true positive).
- The probability of correctly identifying a healthy person is 0.9 (true negative).
If a randomly selected person tests positive, what is the probability that they actually have the disease? (Answer can be given in terms of simplified fraction)

18. Given two normal distributions as $u \sim \mathcal{N}(0, I_2)$ and $v \sim \mathcal{N}(0, 3I_2)$, where $\mathcal{N}(0, I_2)$ is normal distribution with zero mean and covariance matrix as (2×2) identity matrix I_2. Find the determinant of the covariance matrix of z, when $z = u + 2v$ and u, v are independent.

19. You are playing a game, where you start with 1 prize. In each round, you flip a fair coin. If you get a heads, your number of prizes triples, otherwise, you lose all your prizes, and the game ends. You can choose how many times to flip the coin before stopping and keeping your prizes. Let X be the number of coin flips you decide to take. Find the expected number of prizes you have at the end of the game, in terms of X.

20. A fair die is rolled twice. What is the probability that the sum of the two numbers is more than 10?

21. The shortest distance from the curve $xy = 8$ to the origin is

22. Let the function $f : \mathbb{R} \to \mathbb{R}$ be defined as $f(x) = e^{2x+1}$. Then

$$\lim_{x \to 0} \frac{f(f(x)) - f(e)}{x} =$$

23. Evaluate this integral:

$$\int_{-4}^{4} |x - 3| dx$$

24. Evaluate:

$$\lim_{x\to 0} \frac{(1+x)^{\frac{2}{3}}-1}{(1+x)^{\frac{1}{3}}-1}.$$

25. If $S_n = \sum_{k=0}^{n} kp^{k-1}$, where $0 < p < 1$, then $\lim_{n\to\infty} S_n$ is

26. Suppose $f(x, y)$ is defined as

$$f(x,y) = \begin{cases} y, & \text{if } 0 < x < 1, 0 < y < 2, 0 < x + y < 2, \\ 0 & \text{otherwise.} \end{cases}$$

The value of $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y)dxdy$ is

27. The area between the parabolas $y^2 = 2 - x$ and $y^2 = x$ is

28. The directional derivative of $f(x, y, z) = 2x^2 + 3y^2 + z^2$ at a point $P(2, 1, 3)$ in the direction of a vector $\vec{a} = \hat{i} - 2\hat{k}$ is

29. Compute the limit $lim_{x\to\infty}(\sqrt{x^2 + 3x} - x)$

30. Find the global maximum and the value at which it occurs for $f(x) = x^3 - 4x^2 + 5x$ in the range [0 4].

END OF QUESTION PAPER