1. If V and W are 3-dimensional subspaces of $\mathbb{R}^5$, what are all the possible dimensions of $V \cap W$?

2. For what value(s) of the parameter k, will the following system of equations have infinitely many solutions?
$$\begin{aligned} kx + y + z &= 1 \\ x + ky + z &= k \\ x + y + kz &= k^2 \end{aligned}$$

3. Find the product of all the eigenvalues of the matrix $3\boldsymbol{I} + \boldsymbol{A} + \boldsymbol{A}^2$ where $\boldsymbol{I}$ is the identity matrix of dimension 3×3 and
$$\boldsymbol{A} = \begin{pmatrix} 1 & 4 & 6 \\ 0 & 2 & 5 \\ 0 & 0 & 3 \end{pmatrix}$$

4. A real square $n \times n$ matrix $\boldsymbol{A}$ is said to be skew-symmetric if $\boldsymbol{A} = -\boldsymbol{A}^T$. What is the dimension of the vector space of real skew-symmetric matrices of size $n \times n$?

5. Determine all the values of a real number x such that the following matrix is nonsingular.
$$\boldsymbol{A} = \begin{pmatrix} 1 & 1 & -1 \\ 0 & 1 & 2 \\ 1 & 1 & x \end{pmatrix}$$

6. Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be 3×3 real matrices and let $\boldsymbol{C} = \boldsymbol{A} - 3\boldsymbol{B}$. If
$$\boldsymbol{A} \begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix} = \boldsymbol{B} \begin{pmatrix} 3 \\ 9 \\ 15 \end{pmatrix}$$
then what is the minimum dimension of the null space of the matrix $\boldsymbol{C}$?

7. Let $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$ be $n \times n$ invertible real matrices. Simplify the expression
$$\boldsymbol{C}^{-1}(\boldsymbol{A}\boldsymbol{B}^{-1})^{-1}(\boldsymbol{C}\boldsymbol{A}^{-1})^{-1}$$

8. Two eigenvalues of a 3×3 real matrix $\boldsymbol{P}$ are $(2 + \sqrt{-1})$ and 3. What is the value of the sum of all the eigenvalues of $\boldsymbol{P}$?

9. Consider the matrix $\boldsymbol{A} = \boldsymbol{u}\boldsymbol{v}^T$, where $\boldsymbol{u} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $\boldsymbol{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Note $\boldsymbol{v}^T$ denotes the transpose of $\boldsymbol{v}$. If λ_1 and λ_2 are the eigenvalues of $\boldsymbol{A}$ such that $\lambda_1 < \lambda_2$, what is the value of $\lambda_2 - \lambda_1$?

10. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined as
$$f(x, y) = \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

Define the matrix $\boldsymbol{H}$ evaluated at $\begin{pmatrix} a \\ b \end{pmatrix}$ as

$$\boldsymbol{H} = \left. \begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{pmatrix} \right|_{(x=a, y=b)}$$

Find the largest eigenvalue of $\boldsymbol{H}$ evaluated at $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

11. Let $f(x) = a_1 e^{|x|} + a_2 |x|^5$ where a_1 and a_2 are real-valued constants. What are the conditions on a_1 and a_2 if the function $f(\cdot)$ is differentiable at $x = 0$?

12. Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined as

$$f(x) = \frac{(1+x)^2}{1+x^2}$$

At what value of x does the minimum of the function $f(\cdot)$ over the interval [2, 3] occur?

13. Find $\lim\limits_{x \to -\infty} \frac{2x-3}{\sqrt{9x^2+4}}$

14. If $p > 1$, to what value does the integral $\int_1^\infty \frac{dx}{x^p}$ converge? (Hint: $\int_1^\infty \frac{dx}{x^p} = \lim\limits_{b \to \infty} \int_1^b \frac{dx}{x^p}$)

15. Let $f(x) = R \sin \frac{\pi x}{2} + S$. $f'\left(\frac{1}{2}\right) = \sqrt{2}$ and $\int_0^1 f(x)\, dx = \frac{2R}{\pi}$, then what are the values of the constants R and S?

16. Let $f(x, y) = \frac{ax^2 + by^2}{xy}$, where a and b are constants. If $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y}$ at $x = 1$, $y = 2$, then what is the relation between a and b?

17. What is the value of the directional derivative of the field $u(x, y, z) = x^3 - 3yz$ in the direction of the vector $(\frac{1}{3}\hat{i} + \frac{2}{3}\hat{j} - \frac{2}{3}\hat{k})$ at point $(1,\ 1,\ 4)$?

18. What is the value of the integral $\int_0^2 \int_0^x (e^{x+y})\, dy\, dx$?

19. Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined as

$$f(x) = \begin{cases} \max(0, 1-x) & \text{if } x \neq 1 \\ 1 & \text{if } x = 1 \end{cases}$$

Then, what is the value of $\lim\limits_{x \to 1} f(x)$?

20. If $p \in \mathbb{R}$, what is the range of values of p for which the following series is absolutely convergent?

$$\sum_{n=1}^{\infty} \frac{p^n}{n^2}$$

21. Let X and Z be two independent zero mean random variables with second moments $\mathbb{E}[X^2] = \sigma_X^2$ and $\mathbb{E}[Z^2] = \sigma_Z^2$. If $Y = X + Z$, then what is the value of $\mathbb{E}[ZY]$?

22. Let X be a random variable uniformly distributed in $[a, b]$ with $a < b$. Let $Y = 3X+2$ and the cumulative distribution function of Y be $F_Y(y)$. What is the smallest value of y such that $F_Y(y) = 1$?

23. Let X and Y be Bernoulli random variables with $P(X = 0) = 0.5$ and $P(Y = 0) = 0.8$. What is the largest value that $P(X = 1, Y = 1)$ can take?

24. Coin A and Coin B are two biased coins, each when tossed, show heads with probability p and q respectively. One of these coins is chosen at random and tossed 4 times independently. Given that all the four tosses showed tails, what is the probability that Coin A was chosen?

25. India and Australia play a series of **two** cricket matches. The probability of India winning a cricket match is $\frac{1}{2}$, Australia winning is $\frac{1}{6}$ and a no-result (or draw) is $\frac{1}{3}$. The outcomes of each match are independent. India wins the series if it wins more matches than Australia. What is the probability that India wins the series?

26. For a Poisson distribution, $P(X = k) = P(X = k + 1)$ for some positive integer k. Find the mean and variance of this distribution (in terms of k).

27. The probabilities of occurrences of two independent events A and B are 0.5 and 0.8, respectively. What is the probability of occurrence of at least one of the two events A and B?

28. Suppose that a shop has an equal number of LED bulbs of two different types. The probability of an LED bulb lasting more than 100 hours given that it is of Type 1 is 0.7, and given that it is Type 2 is 0.4. What is the probability that an LED bulb chosen uniformly at random lasts more than 100 hours?

29. Suppose X and Y are independent continuous uniform random variables on the interval $[-1, 1]$. What is the variance of $X + Y$?

30. A random variable X has probability density function $f(x)$ as given below:

$$f(x) = \begin{cases} a + bx & \text{if } 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

If the expected value $\mathbb{E}[X] = 2/3$, find the values of a and b.

END OF QUESTION PAPER