1. Find the value(s) of k for which the following system of equations has infinitely many solutions.

$$\begin{aligned}(k+1)x + 8y &= 4k \\ kx + (k+3)y &= 3k-1\end{aligned}$$

2. Let $\mathbf{A}$ be a real 2×2 matrix such that $\mathbf{A} \neq \mathbf{I}$ and $\mathbf{A} \neq -\mathbf{I}$ where $\mathbf{I}$ is the identity matrix of appropriate size. If $\mathbf{A} = \mathbf{A}^{-1}$, then find the trace of $\mathbf{A}^2$.

3. Let $\mathbf{u}$ and $\mathbf{x}$ be n-dimensional vectors such that

$$\mathbf{u} = \frac{1}{\sqrt{n}}\begin{bmatrix}1\\1\\\vdots\\1\end{bmatrix} \quad \mathbf{x} = \begin{bmatrix}1\\2\\\vdots\\n\end{bmatrix}.$$

The value of α such that $\mathbf{x} - \alpha\mathbf{u}$ is orthogonal to $\mathbf{u}$ for $n = 100$ is

4. $\mathbf{A}$ is an $n \times n$ matrix such that the sum of the elements of each column of $\mathbf{A}$ equals 2. If $\mathbf{I}$ denotes an $n \times n$ identity matrix, then find one of the eigenvalues of the matrix $(\mathbf{A} - 3\mathbf{I})$.

5. $\mathbf{A}$ is a 3×3 matrix with eigenvalues as 0, -1 and 1. What is the value of the trace of the matrix $(\mathbf{I} + \mathbf{A}^{100})$, where $\mathbf{I}$ is an identity matrix of appropriate size?

6. If $\mathbf{A}$ is a 5×5 matrix with determinant 2, what is the determinant of the matrix $2\mathbf{A}$?

7. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation such that $T\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = \begin{bmatrix}1\\2\end{bmatrix}$ and $T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \begin{bmatrix}2\\1\end{bmatrix}$. The value of $T\left(\begin{bmatrix}2\\1\end{bmatrix}\right)$ is

8. Find one of the normalized eigenvectors (eigenvector having unit length) of the matrix $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 2 & -2 \end{pmatrix}$.

9. Find all the values of a real number x such that the following three vectors are linearly independent.
$$v_1 = \begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 0 \\ 3 \\ 18x \end{pmatrix} \text{ and } v_3 = \begin{pmatrix} x \\ 0 \\ x+1 \end{pmatrix},$$

10. If V and W are 2-dimensional subspaces of $\mathbb{R}^4$, what are all the possible dimensions of $V \cap W$?

11. A piece of wire of length $4L$ is bent at random to form a rectangle. What is the probability that the area of the rectangle is at most $\frac{L^2}{4}$?

12. Let X denote a continuous random variable which follows Normal or Gaussian distribution with mean 0 and variance 1. Let $Y = aX + b$ where $a > 0$ and $b \in \mathbb{R}$. What is the mean of the random variable Y?

13. Let X and Y be two zero mean random variables such that $\mathbb{E}[X^2] = \sigma_X^2$ and $\mathbb{E}[Y^2] = \sigma_Y^2$ and the correlation coefficient between X and Y is ρ. Then, what is the value of $\mathbb{E}[(X+Y)^2]$?

14. The Indian cricket team's chances of winning the T20 World Cup significantly depend on the physical fitness availability of Hardik Pandya. The probability that he is physically fit is 0.3. The probability that India win the world cup if he is fit is 0.8. The probability that India win the world cup if he is not fit and therefore cannot play is 0.5. Note that if he is fit, he plays. What is the probability that India win the world cup?

15. A and B play a coin tossing game. They toss a fair coin alternatively. The first one to get a Heads wins. What is the probability of A winning if A starts the game?

16. One in two hundred people in a population have a particular disease. A test is developed for the disease. The test gives a false positive 3% of the time (i.e. reports that a person has the disease even when he/she does not), and a false negative 2% of the time (i.e. reports that a person does not have the disease even when he/she does). Bill takes the test and the report comes positive. What is the probability that Bill has the disease?

17. Let X and Y be Bernoulli random variables with $P(X = 0) = 0.5$ and $P(Y = 0) = 0.7$. What is the largest value that $P(X = 1, Y = 1)$ can take?

18. If a random variable X has a Poisson distribution with mean 5, then what is the value of the expectation $\mathbb{E}[(X + 2)^2]$?

19. Two dice are rolled. What is the probability of getting a sum of 9?

20. In a factory, 1% of the products are defective. What is the probability that none of the 3 products chosen for quality control are defective? (assuming independent events)

21. If

$$f'(x) = x(e^x - e)(x - 1)^3(x - 2)^2(x - 3)$$

find the point x^* at which the function $f(x)$ has a minimum.

22. Find the value of the following limit.

$$\lim_{x \to \infty} (x^3 + 6x^2 + 13x + 1)^{\frac{1}{3}} - x$$

23. $\mathbf{A}$ is a 2×2 symmetric matrix with eigen vector $\mathbf{u}$ and corresponding eigen value 0. Let $\mathbf{x}$ be a two-dimensional vector such that the function

$$f(\mathbf{x}) = \mathbf{x}^T \mathbf{A} \mathbf{x}.$$

The gradient of the function is defined as $\nabla f(\mathbf{x}) = \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \end{bmatrix}$. Find an $\mathbf{x}$ for which $\nabla f(\mathbf{x}) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

24. Let $f(x) = Rx + S \cos \pi x$. $f'\left(\frac{1}{2}\right) = 0$, and $\int_0^1 f(x)\, dx = \frac{1}{2}$, then what are the values of the constants R and S?

25. If $|z|$ denotes the absolute value of z, then evaluate the integral:

$$\int_{-1}^{1} |x sin(\pi x)|\, dx$$

26. Evaluate:

$$\lim_{x \to 0} \frac{(1+x)^{\frac{1}{2}} - 1}{(1+x)^{\frac{1}{3}} - 1}.$$

27. Let $f(x, y) = \ln(ax^2 + by^2 + cxy)$, where a, b and c are positive constants. Find the value of $x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y}$ at $x = 2$, $y = 2$.

28. Let R be the region bounded by $y = x^2$ and $y = x$. What is the value of the integral $\iint_R x\, dxdy$?

29. The temperature at a point (x, y) on a metal plate is given by $T(x, y) = x^2 - 2xy + y^2$. Find the rate of change of temperature with respect to x at the point (2, 5).

30. Let $f(x) = a_1|x|^5 + a_2 e^{|x|}$ where a_1 and a_2 are real-valued constants. What are the conditions on a_1 and a_2 if the function $f(\cdot)$ is differentiable at $x = 0$?