1. If V and W are 2-dimensional subspaces of $\mathbb{R}^4$, what are all the possible dimensions of $V \cap W$?

2. For what value(s) of the parameter k, will the following system of equations have no solution?
$$\begin{aligned} kx + y + z &= 1 \\ x + ky + z &= k \\ x + y + kz &= k^2 \end{aligned}$$

3. Find the sum of all the eigenvalues of the matrix $3\boldsymbol{I} + \boldsymbol{A} + \boldsymbol{A}^2$ where $\boldsymbol{I}$ is the identity matrix of dimension 3×3 and
$$\boldsymbol{A} = \begin{pmatrix} 1 & 4 & 6 \\ 0 & 2 & 5 \\ 0 & 0 & 3 \end{pmatrix}$$

4. A real square $n \times n$ matrix $\boldsymbol{A}$ is said to be symmetric if $\boldsymbol{A} = \boldsymbol{A}^T$. What is the dimension of the vector space of real symmetric matrices of size $n \times n$?

5. Determine all the values of a real number x such that the following matrix is nonsingular.
$$\boldsymbol{A} = \begin{pmatrix} 3 & 0 & x \\ 2 & 3 & 0 \\ 0 & 18x & x+1 \end{pmatrix}$$

6. Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be 3×3 real matrices and let $\boldsymbol{C} = \boldsymbol{A} - 2\boldsymbol{B}$. If
$$\boldsymbol{A} \begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix} = \boldsymbol{B} \begin{pmatrix} 2 \\ 6 \\ 10 \end{pmatrix}$$
then what is the maximum possible rank of the matrix $\boldsymbol{C}$?

7. Let $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$ be $n \times n$ invertible real matrices. Simplify the expression
$$\boldsymbol{C}^{-1}(\boldsymbol{A}\boldsymbol{B}^{-1})^{-1}(\boldsymbol{C}\boldsymbol{A}^{-1})^{-1}\boldsymbol{C}^2$$

8. Two eigenvalues of a 3×3 real matrix $\boldsymbol{P}$ are $(2 + \sqrt{-1})$ and 3. What is the value of the product of all the eigenvalues of $\boldsymbol{P}$?

9. Consider the matrix $\boldsymbol{A} = \boldsymbol{u}\boldsymbol{v}^T$, where $\boldsymbol{u} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $\boldsymbol{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Note $\boldsymbol{v}^T$ denotes the transpose of $\boldsymbol{v}$. If λ_1 and λ_2 are the eigenvalues of $\boldsymbol{A}$ such that $\lambda_1 < \lambda_2$, what is the value of $\lambda_1 - \lambda_2$?

10. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined as
$$f(x, y) = \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

Define the matrix $\boldsymbol{H}$ evaluated at $\begin{pmatrix} a \\ b \end{pmatrix}$ as

$$\boldsymbol{H} = \left.\begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{pmatrix}\right|_{(x=a, y=b)}$$

Find the smallest eigenvalue of $\boldsymbol{H}$ evaluated at $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

11. Let $f(x) = a_1|x|^5 + a_2 e^{|x|}$ where a_1 and a_2 are real-valued constants. What are the conditions on a_1 and a_2 if the function $f(\cdot)$ is differentiable at $x = 0$?

12. Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined as

$$f(x) = \frac{(1+x)^2}{1+x^2}$$

At what value of x does the minimum of the function $f(\cdot)$ over the interval [-3, -2] occur?

13. Find $\lim\limits_{x \to -\infty} \frac{3x+2}{\sqrt{4x^2+9}}$

14. If $p > 1$, to what value does the integral $\int_1^\infty \frac{dx}{x^p}$ converge? (Hint: $\int_1^\infty \frac{dx}{x^p} = \lim\limits_{b \to \infty} \int_1^b \frac{dx}{x^p}$)

15. Let $f(x) = Rx + S\cos \pi x$. $f'\left(\frac{1}{2}\right) = 0$, and $\int_0^1 f(x)\,dx = \frac{1}{2}$, then what are the values of the constants R and S?

16. Let $f(x, y) = \ln(ax^2 + by^2 + cxy)$, where a, b and c are positive constants. Find the value of $x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y}$ at $x = 2$, $y = 2$.

17. What is the value of the directional derivative of the field $u(x, y, z) = x^2yz + 4xz$ in the direction of the vector $(\frac{2}{3}\hat{\imath} - \frac{1}{3}\hat{\jmath} - \frac{2}{3}\hat{k})$ at point $(1, 2, 1)$?

18. What is the value of the integral $\int_0^2 \int_0^x (x^2 + y^3)\,dy\,dx$?

19. Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined as

$$f(x) = \begin{cases} \max(1, 1+x) & \text{if } x \neq 0 \\ 2 & \text{if } x = 0 \end{cases}$$

Then, what is the value of $\lim\limits_{x \to 0} f(x)$?

20. If $p \in \mathbb{R}$, what is the range of values of p for which the following series is absolutely convergent?

$$\sum_{n=1}^{\infty} \frac{p^n}{n!}$$

21. Let X and Z be two independent zero mean random variables with second moments $\mathbb{E}[X^2] = \sigma_X^2$ and $\mathbb{E}[Z^2] = \sigma_Z^2$. If $Y = X + Z$, then what is the value of $\mathbb{E}[XY]$?

22. Let X be a random variable uniformly distributed in $[a, b]$ with $a < b$. Let $Y = 2X+3$ and the cumulative distribution function of Y be $F_Y(y)$. What is the smallest value of y such that $F_Y(y) = 1$?

23. Let X and Y be Bernoulli random variables with $P(X = 0) = 0.5$ and $P(Y = 0) = 0.7$. What is the largest value that $P(X = 1, Y = 1)$ can take?

24. Coin A and Coin B are two biased coins, each when tossed, show heads with probability p and q respectively. One of these coins is chosen at random and tossed 5 times independently. Given that all the five tosses showed tails, what is the probability that Coin A was chosen?

25. India and Australia play a series of **two** cricket matches. The probability of India winning a cricket match is $\frac{1}{2}$, Australia winning is $\frac{1}{3}$ and a no-result (or draw) is $\frac{1}{6}$. The outcomes of each match are independent. India wins the series if it wins more matches than Australia. What is the probability that India wins the series?

26. If a random variable X has a Poisson distribution with mean 5, then what is the value of the expectation $\mathbb{E}[(X + 2)^2]$?

27. The probabilities of occurrences of two independent events A and B are 0.3 and 0.8, respectively. What is the probability of occurrence of at least one of the two events A and B?

28. Suppose that a shop has an equal number of LED bulbs of two different types. The probability of an LED bulb lasting more than 100 hours given that it is of Type 1 is 0.5, and given that it is Type 2 is 0.9. What is the probability that an LED bulb chosen uniformly at random lasts more than 100 hours?

29. Suppose X and Y are independent continuous uniform random variables on the interval $[0, 1]$. What is the variance of $X + Y$?

30. A random variable X has probability density function $f(x)$ as given below:

$$f(x) = \begin{cases} a + bx & \text{if } 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

If the expected value $\mathbb{E}[X] = 2/3$, find the probability $P\left(X < 0.5\right)$.

END OF QUESTION PAPER